On the Combination of the different Results of various Series of Observations. By Dr. C. Powalky.

1. *Direct solution* for two series.

Let $[v_1]$ be the sum of all values v_1 of the first series; n_1 the number of its observations; $[d_1 d_1]$ the sum of the squares, $v_1 - \frac{[v_1]}{n_1}$;

Let $[v_2]$ be the sum of all values v_2 of the second series; n_2 the number of its observations; $[d_2 d_2]$ the sum of the squares, $v_2 - \frac{[v_2]}{n_2}$;

p_1, p_2 the weights.

The general equation is:

$$w = \frac{p_1[v_1]+p_2[v_2]}{p_1 n_1 + p_2 n_2} = \frac{[v_1]}{n_1} - \frac{\left(\frac{[v_1]}{n_1} - \frac{[v_2]}{n_2}\right)p_2 n_2}{p_1 n_1 + p_2 n_2} = \frac{[v_2]}{n_2} + \frac{\left(\frac{[v_1]}{n_1} - \frac{[v_2]}{n_2}\right)p_1 n_1}{p_1 n_1 + p_2 n_2}.$$

Putting now

$$\frac{[v_1]}{n_1} - \frac{[v_2]}{n_2} = \mathrm{D};\ p_1 n_1 + p_2 n_2 = \mathrm{Z},$$

the squares of the errors are

$$[d_1 d_1] + n_1 \frac{\mathrm{D}^2 n_2^2 p_2^2}{\mathrm{Z}^2};\ [d_2 d_2] + n_2 \frac{\mathrm{D}^2 n_1^2 p_1^2}{\mathrm{Z}^2};$$

and therefore

$$p_1 : p_2 = \frac{[d_2 d_2] + n_2 \frac{\mathrm{D}^2 n_1^2 p_1^2}{\mathrm{Z}^2}}{n_2 - 1} : \frac{[d_1 d_1] + n_1 \frac{\mathrm{D}^2 n_2^2 p_2^2}{\mathrm{Z}^2}}{n_1 - 1}$$

$$= (n_1 - 1)([d_2 d_2]\,\mathrm{Z}^2 + n_2 n_1^2 \mathrm{D}^2 p_1^2) : (n_2 - 1)([d_1 d_1]\,\mathrm{Z}^2 + n_1 n_2^2 \mathrm{D}^2 p_2^2),$$

or

$$(n_2 - 1)p_1\{\mathrm{Z}^2 [d_1 d_1] + n_1 n_2^2 \mathrm{D}^2 p_2\} = (n_1 - 1)p_2\{\mathrm{Z}^2 [d_2 d_2] + n_1^2 n_2 \mathrm{D}^2 p_1\}.$$

Taking $p_1 = q p_2$, we have

$$\mathrm{Z}^2 = p_2^2 (q n_1 + n_2)^2, \text{ and dividing by } p_2^3$$

$$(n_2 - 1)q\{(q n_1 + n_2)^2 [d_1 d_1] + n_1 n_2^2 \mathrm{D}^2\} = (n_1 - 1)\{(q n_1 + n_2)^2 [d_2 d_2] + n_1^2 n_2 \mathrm{D}^2\};$$

or

$$\alpha q^3 + \beta q^2 + \gamma q + \delta = 0.$$

If the problem is determinate, we find only one real value of q

from this equation, the other two being imaginary; but the problem is indeterminate if there are three real values.

2. The *indirect solution* is more practicable, and applicable to any number of series.

Take w as known, we have for the determination of $p_1, p_2, p_3 \ldots$

$$\frac{1}{p_1} = \left\{\left(w - \frac{[v_1]}{n_1}\right)^1 n_1 + [d_1 d_1]\right\} \div (n_1 - 1)$$

$$\frac{1}{p_2} = \left\{\left(w - \frac{[v_2]}{n_2}\right)^2 n_2 \div [d_2 d_2]\right\} \div (n_2 - 1)$$

$$\frac{1}{p_3} = \left\{\left(w - \frac{[v_3]}{n_3}\right)^3 n^3 + [d_3 d_3]\right\} \div (n_3 - 1).$$

$$\vdots \qquad\qquad\qquad \vdots$$

Assuming for w any value for w', determine the weights for this, and determine—if w' is not right—another value w''; assuming now $w = w''$, determine a new value w''', and so on until $w^{(n+1)} = w^{(n)}$.

Example.

$\frac{[v_1]}{n_1} = +12{\cdot}8,\ n_1 = 9;\ \frac{[v_2]}{n_2} = +1{\cdot}5,\ n_2 = 4;\ \frac{[v_3]}{n_3} = +1{\cdot}6,\ n_3 = 5;\ \frac{[v_4]}{n_4} = \div 0{\cdot}5,\ n_4 = 7.$

$[d_1 d_1] = 40{\cdot}0 \qquad [d_2 d_2] = 1{\cdot}1 \qquad [d_3 d_3] = 12{\cdot}3 \qquad [d_4 d_4] = 12{\cdot}0.$

1. $w' = +6{\cdot}0.$

n	9	4	5	7
$w - \frac{[v]}{n}$ . . .	−6·8	+4·5	+4·4	+5·5
Squares . . .	46·24	20·25	19·37	30·26
Multip. by n . .	416·2	81·0	96·9	211·8
+ $[dd]$ . . .	40·0	1·1	12·3	12·0
Sum . . .	456·2	82·1	109·2	223·8
÷ by $(n-1)$ . .	57·0	27·4	27·3	37·3
$\log \frac{1}{p} =$ . . .	1·7559	1·4378	1·4362	1·5717
$\log [v]$ . . .	2·0615	0·7782	0·9031	0·5441
$\log p$	8·2441	8·5622	8·5638	8·4283
$\log n$	0·9542	0·6021	0·6990	0·8451

$[v]p$	np
+2·021	0·158
+0·219	0·146
+0·293	0·183
+0·094	0·188
+2·627	0·675

$w'' = +3{\cdot}9$

$$w'' = \frac{[v_1]p_1 + [v_2]p_2 + [v_3]p_3 + [v_4]p_4}{n_1 p_1 + n_2 p_2 + n_3 p_3 + n_4 p_4}$$

2. $w'' = +3{\cdot}9$.

						[v] p	n p
n	9	4	5	7		+ 1·222	0·096
$w - \frac{[v]}{n}$ . . .	−8·9	+2·4	+2·3	+3·9		+0·747	0·498
Squares . . .	79·22	5·76	5·29	11·56		+0·825	0·516
Multip. by n .	713·0	23·0	26·5	80·9		+0·226	0·451
+ [d d] . . .	40·0	1·1	12·3	12·0		+3·020	1·561
Sum . . .	753·0	24·1	38·8	92·9			
÷ by $(n-1)$ . .	94·3	8·07	9·7	15·5		$w''' = +1{\cdot}93$	
$\log \frac{1}{p}$ = . . .	1·9743	0·9049	0·9868	1·1903			
log [v]	2.0615	0·7782	0·9831	0·5441			
log p	8·0257	9·0951	9·0132	8·8097			
log n	0·9542	0·6021	0·6990	0·8451			

3. $w''' = +1{\cdot}93$.

					[v] p	n p
$w - \frac{[v]}{n}$ = Q .	−10·87	+0·43	+0·33	+1·43	+ 0·835	0·065
Q^2	118·1	0·185	0·109	2·045	+ 10·000	6·667
$n Q^2$	1062·9	0·74	0·54	14·3	+ 2·500	1·563
+ [d d] . . .	40·0	1·1	12·3	12·0	+ 0·760	1·522
Sum = S . .	1102·9	1·84	12·84	26·3	+ 14·095	9·817
$\frac{S}{n-1}$	137·9	0·6	3·2	4·6	$w^{(4)} = +1{\cdot}44$	
log p = . . .	7·8604	0·2218	9·4949	9·3372		

4. $w^{(4)} = +1{\cdot}4$.

					[v] p	n p
$w - \frac{[v]}{n}$ = Q .	−11·4	−0·1	−0·2	+0·9	+ 0·76	0·06
Q^2	130·0	0·01	0·04	0·81	+ 16·22	10·81
$n Q^2$	1170·0	0·04	0·20	5·67	+ 2·58	1·61
+ [d d] . . .	40·0	1·1	12·3	12·0	+ 1·17	2·33
Sum = S . .	1210·0	1·14	12·5	17·67	+ 20·74	24·81
$\frac{S}{n-1}$	151·0	0·38	3·18	2·94	$w^{(5)} = +1{\cdot}40$	
log p = . . .	7·8210	0·4318	9·5086	9·5229	$p_1 = 0{\cdot}007$	
					$p_2 = 2{\cdot}703$	
					$p_3 = 0{\cdot}323$	
					$p_4 = 0{\cdot}333$	

$w = +1{\cdot}40$ is here but one of the values; for, assuming first $w' = +11{\cdot}0$, we find $w'' = +11{\cdot}6$, $w''' = +11{\cdot}9$, and $w^{(4)} = w^{(5)} = +12{\cdot}0$ with the weights: $p_1 = 0{\cdot}170$, $p_2 = 0{\cdot}007$, $p_4 = 0{\cdot}063$. A third value of w will be found $= +9{\cdot}0$, and the weights $p_1 = 0{\cdot}047$, $p_2 = 0{\cdot}013$, $p_3 = 0{\cdot}014$, $p_4 = 0{\cdot}012$. The problem is indeterminate.

Excluding the first value of $\frac{[v]}{n}$ we obtain $w = +1{\cdot}38$, and no other value satisfying the conditions

$$w = \frac{p_2[v_2] + p_3[v_3] + p_4[v_4]}{n_2 p_2 + n_3 p_3 + n_4 p_4},$$

the weights are now:

$$p_2 = 3{\cdot}7,\ p_3 = 0{\cdot}32,\ p_4 = 0{\cdot}34.$$

The problem is determinate.

Washington, 1874, *May*.

Remark on the Influence of Errors of Observation on the Determination of the Orbit of a Planet from three Observations. By Herr F. W. Berg.

(*Translation.*)

Let a be the semiaxis major of the orbit, e ($= \sin\phi$) the excentricity, c the mean anomaly of the epoch, ϖ the distance of perihelion from ascending node, Ω the longitude of ascending node, i the inclination to the ecliptic; and moreover let α and β be the longitude and latitude for the first observation, α' and β' for the second, and α'' and β'' for the third. Then, as is known, compare Gauss, *Theoria Motus*, Art. 76–76:

$$\begin{aligned} d\alpha &= \mathrm{A}\,da + \mathrm{E}\,de + \mathrm{C}\,dc + \mathrm{W}\,d\varpi + \mathrm{I}\,di + \mathrm{O}\,d\Omega, \\ d\beta &= \mathrm{a}\,da + \mathrm{e}\,de + \mathrm{c}\,dc + \mathrm{w}\,d\varpi + \mathrm{i}\,di + \mathrm{o}\,d\Omega, \end{aligned} \tag{1}$$

and similarly for the other two observations.

The coefficients A, E, &c., depend on the time t and the elements a, e, &c., of the orbit, and we have thus in these equations the relation between the variations of α and β and those of the elements. From the equations (1) and the corresponding equations for the other two observations can be obtained new equations serving to express the variations of the elements in terms of the variations $d\alpha$, $d\beta$, $d\alpha'$, $d\beta'$, $d\alpha''$, $d\beta''$. We obtain

$$\begin{aligned} \Delta\,da &= (\alpha, a)\,d\alpha + (\alpha', a)\,d\alpha' + (\alpha'', a)\,d\alpha'' + (\beta, a)\,d\beta + (\beta', a)\,d\beta' + (\beta'', a)\,d\beta'', \\ \Delta\,de &= (\alpha, e)\,d\alpha + (\alpha', e)\,d\alpha' + (\alpha'', e)\,d\alpha'' + (\beta, e)\,d\beta + (\beta', e)\,d\beta' + (\beta'', e)\,d\beta'. \end{aligned} \tag{2}$$

&c.

Considering now the quantities $d\alpha$, $d\beta$, &c., as errors of